zero-to-hero

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Invalid Date

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Welcome

This course is designed to refresh your knowledge of maths to get you ready to use calculus in your course. There is no right or wrong way to use it. Each section includes written notes, a video (with the same content as the notes) and practice questions. It's chunked into bitesized sections to allow you make progress in 10 min windows. You may like to try the questions first and then just go back to the notes if you get stuck. Feel free to start anywhere you like.

> ⚠ Warning
>
> The videos are hosted on the University's Panopto Re:view server. You will have to login to watch them - it may also force a pop-up window.

This is a work in progress, the videos are appearing and things may change! If you find a mistake please email edrs20@bath.ac.uk and good luck!

> Let me know what you think
>
> If you've got some spare time to fill in this survey about this resource, I'd love to know what you think of it.

1 Negative numbers

On a number line negative numbers are typically written to the left of zero and have values smaller than zero. Negative numbers are tricky. Often when an error creeps into a calculation it's due to a misplaced minus sign, they are a source of problems for everyone - don't worry if they seem tricky, they have only relatively recently lost their mysteriousness. The evidence of humans counting dates from $35,000$BCE yet as recently as 1758 British mathematician Francis Maseres said that negative numbers…

"… darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple".

1.1 Multiplication and Division

When multiplying and dividing using negative numbers the answer will be the same as the equivalent calculation with positive numbers only, but, you may have to change the sign - to either positive or negative. The rules for deciding if the answer is positive or negative are below:

> **Note**
>
> - positive $\times$ positive $=$ positive
> - negative $\times$ positive $=$ negative
> - positive $\times$ negative $=$ negative
> - negative $\times$ negative $=$ positive

Notice that the order is not important. Here are some examples:

$$-2 \times 3 = -6$$

$$10 \times -5 = -50$$

$$-4 \times -6 = 24$$

If you have more that two numbers to multiplying you can just count the number of negative numbers and apply the following rule:

Note

- If the total number of negative numbers is **even** the answer is **positive**.
- If the total number of negative numbers is **odd** the answer is **negative**.

Here's a longer example:

$$-2 \times -2 \times -2 \times -2 = 16$$

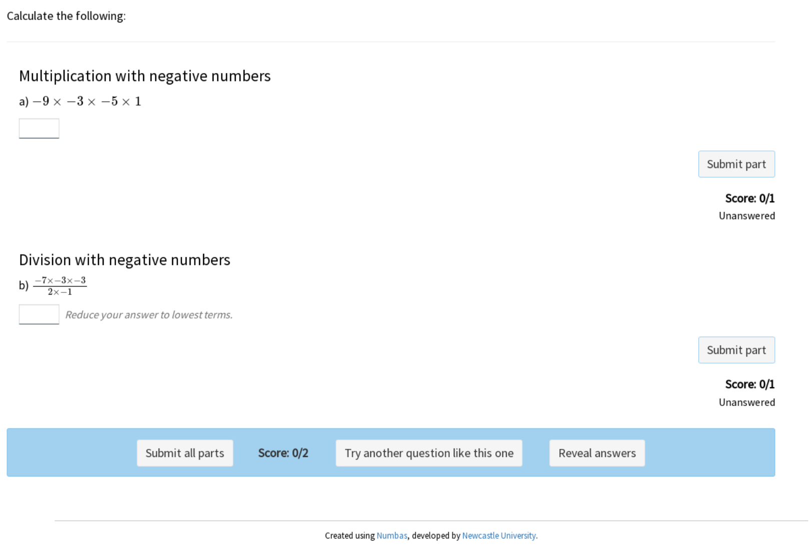
since there are even number of negatives in the question the answer will be positive.

Since division and multiplication are so closely related, division works in exactly the same way. For example:

$$\frac{-3 \times -6}{-9} = -2$$

.

You can practice these techniques with the following questions. You can refresh the question to change the numbers. Try them as much as you like.



1.1.1 But why?!!?

Building a physical idea of a negative number is tricky. For example thinking of 2×3 as two lots of 3 things is fine, but what does -2×-3 even mean? Hopefully but looking at the pattern below it will be become clear that our definition of what happens with two negative numbers is the only one that makes sense. Consider extending the two times table into negative numbers.

$$\begin{array}{rcrcr} 3 & \times & 2 & = & 6 \\ 2 & \times & 2 & = & 4 \\ 1 & \times & 2 & = & 2 \\ 0 & \times & 2 & = & 0 \\ -1 & \times & 2 & = & -2 \\ -2 & \times & 2 & = & -4 \\ -3 & \times & 2 & = & -6 \end{array}$$

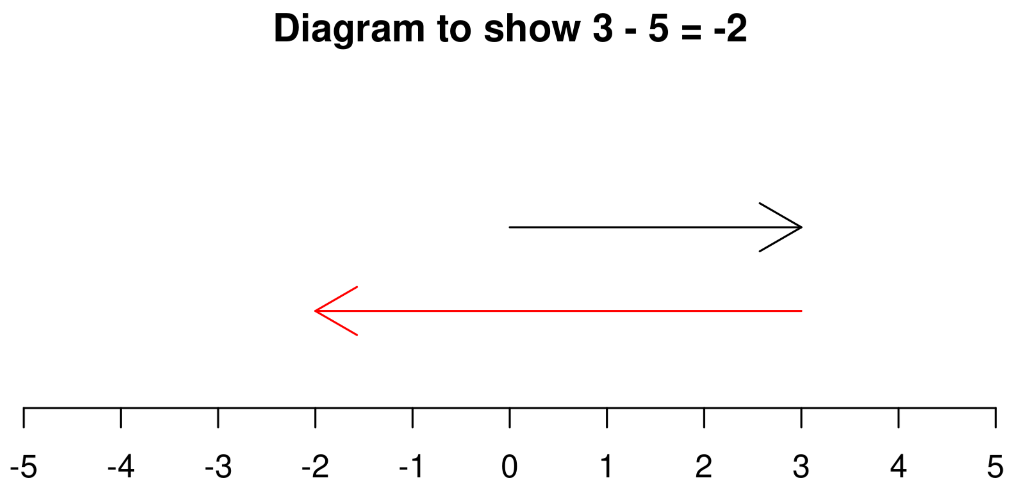
Now with the negative two times table.

$$\begin{array}{rcrcr} 3 & \times & -2 & = & -6 \\ 2 & \times & -2 & = & -4 \\ 1 & \times & -2 & = & -2 \\ 0 & \times & -2 & = & 0 \\ -1 & \times & -2 & = & 2 \\ -2 & \times & -2 & = & 4 \\ -3 & \times & -2 & = & 6 \end{array}$$

Our definition fits the pattern. Horrah!

1.2 Addition and subtraction

It helps to think about addition and subtraction of negative numbers on a number line. We can think about positive numbers as arrows pointing *forwards*, shifts to the right from zero, and negative numbers as arrows *backwards*, shifts to the left. Add to this the idea that addition and subtraction is then combining these arrows. When you add two numbers you place them one after another, the end of the second arrow on the tip of the first. With subtraction you reverse the direction of the second arrow and then place them together just like addition.



Consider the following examples:

- $3 - 5 = -2$ can be thought of as: start at three then move five back to the left.
- $-4 + 1 = -3$, start at -4 then move one to the right.
- $5 + -2 = 3$, start at 5 then add on a shift of 2 to the left.
- $1 - -3 = 4$, start at 1 then reverse a shift of 3 to the left (I know it seams bonkers!). The double negative cancels out to give a calculation equivalent to $1 + 3 = 4$.

> **Warning**
>
> It's tempting to cling on to the idea that *two negatives make a positive* when it comes to addition and subtraction. But consider the following statements, they are all correct, but imagine how easy it is to be confused if you just apply the *two negatives make a positive* rule.
>
> - $-3 - 5 = -8$
> - $-10 - -4 = -6$

You can practice these techniques with the following questions. The numbers change each time to try them as much as you like.

Calculate:

a)

$-7 + -2 =$

Submit part

Score: 0/1

Unanswered

b)

$-6 - -6$

Submit part

Score: 0/1

Unanswered

Submit all parts **Score: 0/2** Try another question like this one Reveal answers

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2 Algebraic expressions

Algebraic expressions are just statements about numbers. However, letters are used as place holders for some of the numbers. There are many reasons this is useful, it could be because we would like to uncover the structure of something, or, because we don't know the specific numbers to use yet.

2.1 Substitution

In order to evaluate an algebraic expression we have to substitute the letters for numbers. After the numbers are written in place of the letters we must take care to evaluate the statement in the correct order. BIDMAS is often used to remember the order:

- **Brackets** Work out anything in brackets first.
- **Indices** Powers are next, something like 3^2.
- **Division and Multiplication** these two have equal priority. If there is a 'tie' work left to right. However if you see a large division they have implicit brackets in them. For example $\frac{2+10}{2\times 3}$ should be thought of as $\frac{(2+10)}{(2\times 3)}$.
- **Addition and Subtraction** like multiplication and division these are equal priority. If there is a tie work left to right.

One more thing to know before we start making substitutions is that the multiplication symbol $\times$ is often not used in algebraic expressions. Letters and numbers that are next to each other are multiplied together. For example $3a$ means $3 \times a$. You can show two numbers multiplied together like this $2 \times 3 = (2)(3) = 6$.

Here are some examples:

If $a = 2$ and $b = -3$ then we can evaluate $5a + 4b$ like this:

$$5(2) + 4(-3)$$

When things are written next to each other this means multiplication.

$$5 \times 2 + 4 \times -3$$

Using BIDMAS to do the multiplication first and remembering that a positive number multiplied by a negative gives a negative number.

$$10 + -12 = -2$$

Substituting $n = 3$ and $x = 2$ into $5x^n$. By replacing the letters with numbers we have:

$$5(2)^3$$

Remembering that when things are next to each other it means multiplication, which gives:

$$5 \times 2^3$$

Following BIDMAS we must deal with the powers first. Since $2^3 = 2 \times 2 \times 2 = 8$ we have:

$$5 \times 8 = 40$$

Finally consider $\frac{2p+q}{r}$ where $p = 6$, $q = 3$ and $r = 5$. Replacing the letters with numbers we have:

$$\frac{2(6)+3}{5} = \frac{2 \times 6 + 3}{5}$$

Remembering that there are implicit brackets in fractions, the numerator needs to be evaluated first.

$$\frac{(2 \times 6 + 3)}{5} = \frac{(12+3)}{5} = \frac{15}{5}$$

Now the fraction can be evaluated.

$$\frac{15}{5} = 3$$

You can practice these techniques with the following questions. The numbers change each time to try them as much as you like.

2.2 Simplification

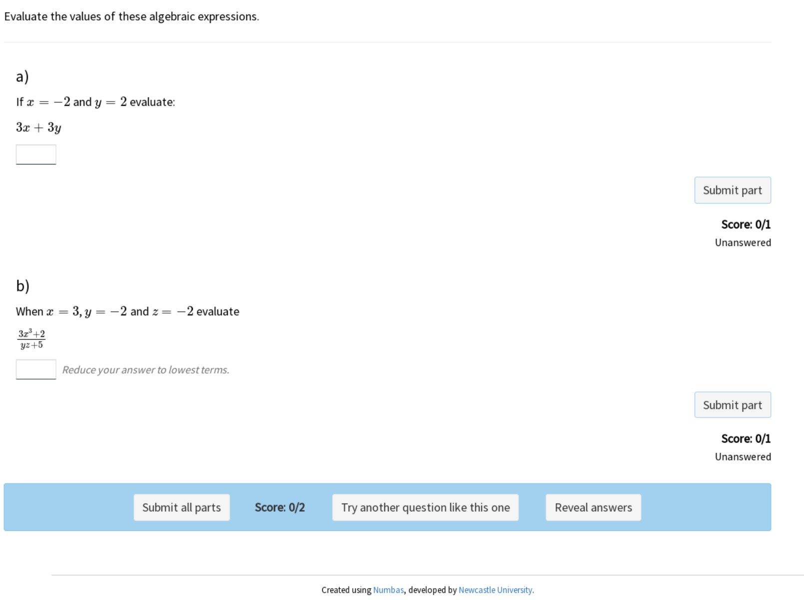
Algebraic expressions are made up of terms. Similar terms can be combined to create a simplified expression, this processes is called *collecting like terms*. For example $2a + 3a$ can be simplified to $5a$ by collecting the a terms. Here's another example with a bit more going on:

$$5x + 7y - 3x + 3y = \overbrace{5x - 3x}^{x \text{ terms}} + \overbrace{7y + 3y}^{y \text{ terms}} = 2x + 10y$$

Notice that the like terms were grouped first to make it easier to simplify. Also, each term *owns* the positive or negative symbol ahead of it.

Terms can be more complex too. Although it's tempting to find something to simplify there are no like terms in this expression: $3xy + 6x^2 + 2x - 5y$. Only the exact same multiples can be simplified. For example:

$$6x^2 + 2x - 5x^2 - 8x = \overbrace{6x^2 - 5x^2}^{x^2 \text{ terms}} + \overbrace{2x - 8x}^{x \text{ terms}} = x^2 - 6x$$



Notice that the two different types of term are x and x^2. Also, I could have written $1x^2$ but we normally don't bother with the 1. It's also important to note that capitalisation matters; x is different from X.

Take care when simplifying multiples of different letters $3xy+5yx$ can be simplified. This is because the order of multiplication doesn't matter so $3xy + 5yx = 3xy + 5xy = 8xy$. Terms are normally written in alphabetical order with the highest powers first.

Key point:

- $x \times x = x^2$
- $x + x = 2x$
- x is different from X
- $1x$ is written as x

Have a go at simplifying with these questions.

For each expression below, simplify by collecting the like terms.

a)

$-5x + 4x + 3x =$

Submit part

Score: 0/1

Unanswered

b)

$-3x^2 - 4 + 9x + 5x + 9x^2 =$

Submit part

Score: 0/1

Unanswered

Submit all parts **Score: 0/2** Try another question like this one Reveal answers

3 Expressions with brackets

Dealing with algebraic expressions containing brackets is a useful skill. This section looks at removing brackets by *expanding* and adding brackets back in by *factorising*.

3.1 Expanding

3.1.1 Single brackets

Expanding a bracket in an algebraic expression is an example of the distributive law. You probably are already familiar with that law. Here is an example of how the law could be used to work out 6×15 using a mental method.

$$\begin{aligned} 6 \times 15 &= 6 \times (10 + 5) \\ &= 6 \times 10 + 6 \times 5 \\ &= 60 + 30 \\ &= 90 \end{aligned}$$

The same procedure is followed with an algebraic expression.

$$\begin{aligned} 6(2x + 5) &= 6 \times (2x + 5) \\ &= 6 \times 2x + 6 \times 5 \\ &= 12x + 30 \end{aligned}$$

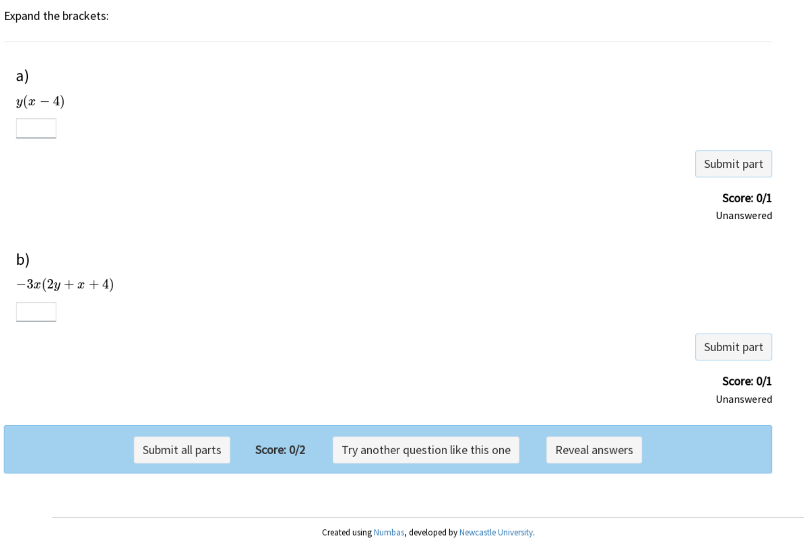
The number of terms within the bracket isn't limited to two. For example:

$$\begin{aligned} x(y + 3x - 5) &= x \times (y + 3x - 5) \\ &= x \times y + x \times 3x + x \times -5 \\ &= xy + 3x^2 - 5x \end{aligned}$$

Finally, another common pattern is to have a negative sign before a bracket. This just means everything inside the bracket is multiplied by -1. It just *flips* the sign of everything in the brackets.

$$\begin{aligned} -(3-x) &= -1 \times (3-x) \\ &= -1 \times 3 + -1 \times -x \\ &= -3 + x \end{aligned}$$

Here are some practice questions.



3.1.2 Expanding pairs of brackets

This will be covered in Quadratics.

3.2 Factorising

The reverse of expanding brackets is called factorising. We look for a common factor in each term to take outside of the bracket.

3.2.1 Factorising - single brackets

For each term in the expression look for a common factor. We can then write this in front of the bracket so when you expand the bracket the original expression is returned. For example:

$$\begin{aligned} 12x^2 - 15x &= 3x \times 4x + 3x \times -5 \\ &= 3x(4x - 5) \end{aligned}$$

Notice that $3x$ is a factor of both $12x^2$ and $-15x$. Also, if we expand our answer we should get back to where we started from.

Here are some practice questions.


Factorise:

Note: you must enter a multiplication sign between the factor and the bracket type `x*(x+1)` to get $x \times (x + 1)$.

a)

$3xy - 9x$

Submit part

Score: 0/1

Unanswered

b)

$6x^2 + 2xy - 4x$

Submit part

Score: 0/1

Unanswered

Submit all parts **Score: 0/2** Try another question like this one Reveal answers

Created using Numbas, developed by Newcastle University.

3.2.2 Factorising - pairs of brackets

This will be covered in the Quadratics section.

4 Fractions

Fractions can be written in two ways:

- as decimals fractions, for example 0.5, 0.25 and $0.\dot{3}$.
- as vulgar fractions, the following fractions have the same values as the examples above, $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{3}$. Vulgar fractions consist of two parts. The top, or **numerator**, and the bottom, the **denominator**.

Vulgar fractions are useful in algebra. The next section looks at some techniques for dealing with them.

4.1 Simplifying

Fractions can be *cancelled down* or simplified by dividing the numerator and denominator by a common factor i.e. we look for a number that *goes into* both the top and the bottom of the fraction. For example:

$$\begin{aligned}\frac{18}{24} &= \frac{3\times 6}{4\times 6}\\ &= \frac{3\times \cancel{6}}{4\times \cancel{6}}\\ &= \frac{3}{4}\end{aligned}$$

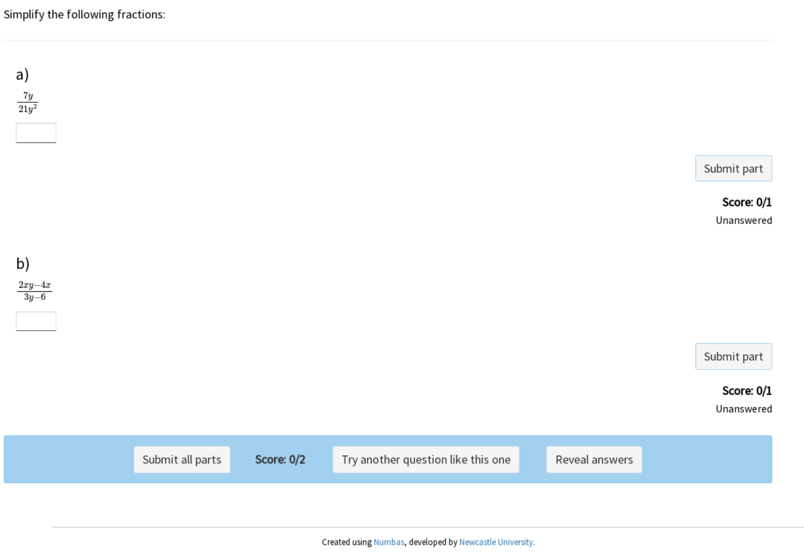
The same can be done with algebraic fractions.

$$\begin{aligned}\frac{4xy}{6x} &= \frac{2y\times 2x}{3\times 2x}\\ &= \frac{2y\times \cancel{2x}}{3\times \cancel{2x}}\\ &= \frac{2y}{3}\end{aligned}$$

Sometimes you'll need to factorise expressions in the fraction in order to cancel it down.

$$\begin{aligned}\frac{10x^2+5x}{4x+2} &= \frac{5x\times 2x+5x\times 1}{2\times 2x+2\times 1}\\ &= \frac{5x(2x+1)}{2(2x+1)}\\ &= \frac{5x\cancel{(2x+1)}}{2\cancel{(2x+1)}}\\ &= \frac{5x}{2}\end{aligned}$$

Here are some practice questions.



! Warning!

It is tempting to want to make cancellations like this:

$$\begin{aligned}\frac{2x^2}{3x+7} &= \frac{2x\cancel{x}}{3\cancel{x}+1}\\ &= \frac{2x}{3+7}\\ &= \frac{2x}{10}\\ &= \frac{x}{5}\end{aligned}$$

However, please don't do it, as it's just plain wrong! Lets let $x = 3$ and substitute it into the original $\frac{2x}{3x+7}$ and into incorrectly simplified version $\frac{x}{5}$. If the algebra is correct it should give the same answer.
We claim:

$$\frac{2x^2}{3x+7} = \frac{x}{5}$$

but if we substitute $x = 2$ into both sides we get:

$$\begin{aligned}\frac{2(3)^2}{3(3)+7} &= \frac{(3)}{5}\\ \frac{2\times 9}{9+7} &= \frac{3}{5}\\ \frac{18}{16} &= \frac{3}{5}\\ \frac{9}{8} &= \frac{3}{5}\end{aligned}$$

Which is nonsense!

4.2 Multiplication and division

Multiplication and division of fractions is, thankfully, really easy!

4.2.1 Multiplication

For multiplication you simply multiply the different fractions numerators and denominators together. In other words the top of the first fraction with the top of the second one and so on. After the multiplication you may be able to cancel down the fraction. Just like this:

$$\begin{aligned}\frac{2}{5}\times\frac{3}{4} &= \frac{2\times 3}{5\times 4}\\ &= \frac{6}{20}\\ &= \frac{3\times 2}{10\times 2}\\ &= \frac{3\times \cancel{2}}{10\times \cancel{2}}\\ &= \frac{3}{10}\end{aligned}$$

Pro-tip

It is possible to cancel before multiplying. Here is the same example revisited:

$$\begin{aligned}\frac{2}{5} \times \frac{3}{4} &= \frac{2 \times 3}{5 \times 4} \\ &= \frac{2 \times 3}{5 \times 2 \times 2} \\ &= \frac{\cancel{2} \times 3}{5 \times 2 \times \cancel{2}} \\ &= \frac{3}{10}\end{aligned}$$

This can be super useful when dealing with large numbers or complex algebraic fractions.

4.2.2 Division

We can change a division into a multiplication by remembering **keep, change, flip**. We keep the first fraction as it is. Change the division, $\div$, symbol to a multiplication, $\times$, and flip the last fraction - swap the places of the numerator and denominator. This is called taking the reciprocal of the fraction. For example:

$$\begin{aligned}\frac{3}{7} \div \frac{5}{2} &= \frac{3}{7} \times \frac{2}{5} \\ &= \frac{3 \times 2}{7 \times 5} \\ &= \frac{6}{35}\end{aligned}$$

4.3 Addition and subtraction

Addition and subtraction is easy if the denominators are the same. We just add the numerators together and the denominators stays the same. For example:

$$\begin{aligned}\frac{2}{5} + \frac{1}{5} &= \frac{2+1}{5} \\ &= \frac{3}{5}\end{aligned}$$

If the denominators are different we must make equivalent fractions with a common denominators first. Finding a common denominator is like simplification, or cancelling down, but in reverse.

If we want to add $\frac{2}{3}$ and $\frac{2}{9}$ for example, we want to rewrite the first fraction so that it has 9 as the denominator. To do this, we multiply the top and bottom of the fraction by 3 (Remember to multiply **both** the numerator and denominator by 3 to make sure the fractions are equivalent!) :

$$\begin{aligned}\frac{2}{3}+\frac{2}{9} &= \frac{2\times 3}{3\times 3}+\frac{2}{9}\\ &= \frac{6}{9}+\frac{2}{9}\\ &= \frac{6+2}{9}\\ &= \frac{8}{9}\end{aligned}$$

5 Solving equations

When we work out the value of an unknown, say x, in an equation we say that we *are solving for* x. To work out the value we are free to apply any mathematical operation we like to the equation so long as we *do the same to both sides*.

Note: We can't quite do any operation. Division by zero, $\div 0$, is not allowed as it is undefined.

5.1 Linear equations

5.1.1 Single unknown

Keeping the idea of doing the same thing to both sides in mind lets solve the following equation by *undoing* each operation with it's inverse.

$$3x + 8 = 10$$

First subtract 8 from each side.

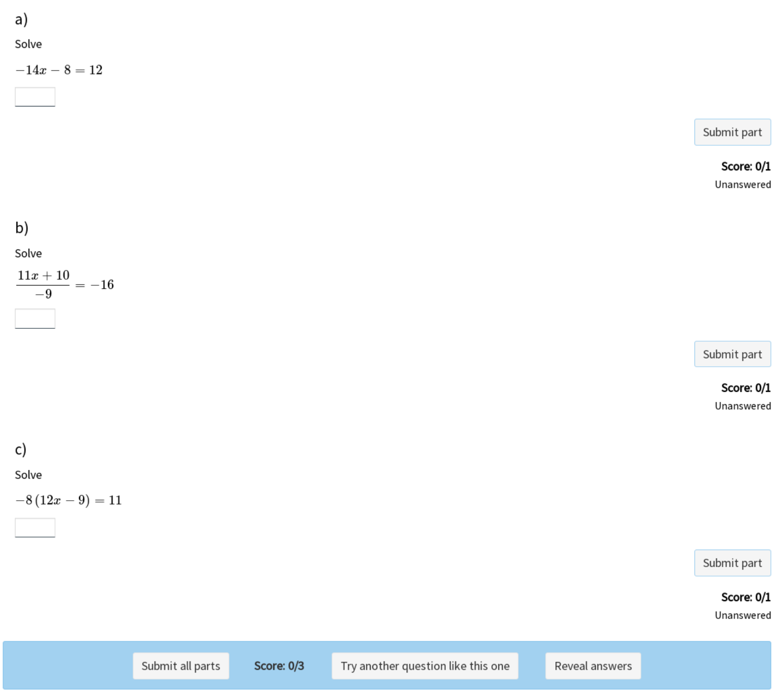
$$\begin{aligned} 3x + 8 - 8 &= 10 - 8 \\ 3x &= 2 \end{aligned}$$

Now divide both sides by 3 to find the value of one x.

$$\begin{aligned} \frac{3x}{3} &= \frac{2}{3} \\ x &= \frac{2}{3} \end{aligned}$$

The nice thing here is that we can leave the answer as $\frac{2}{3}$. No need to find a decimal fraction if we don't need to.

Solve the following equations by applying the same operation to both sides. Remember the questions come with full solutions, so, if you get stuck have a look at the answers and then try a different one.



5.1.2 Unknown on both sides

If the unknown appears twice in an equation collect the unknown like terms first and then solve as before.

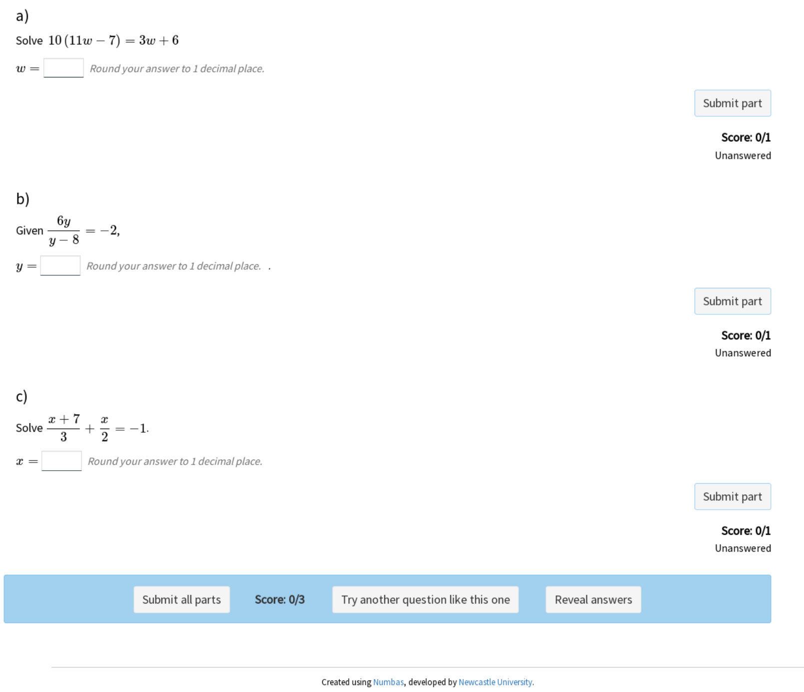
Given $\frac{4y}{y-9} = -2$, we can multiply both sides by $(y - 9)$ to get rid of the fraction, then get all the ys on one side, then finally solve as before.

$$\begin{aligned}
\frac{4y}{y-9} &= -2 \\
\frac{4y}{y-9} \times (y-9) &= -2 \times (y-9) \\
\frac{4y}{\cancel{y-9}} \times \cancel{(y-9)} &= -2 \times y - 2 \times -9 \\
4y &= -2y + 18 \\
4y + 2y &= -2y + 18 + 2y \\
6y &= 18 \\
\frac{6y}{6} &= \frac{18}{6} \\
y &= 3
\end{aligned}$$

Note

- To solve equations do the same thing to both sides.
- If the unknown appears twice - collect like terms first.

Have a go at some questions. You'll need a pen and paper to work these out.



5.2 Inequalities

Solving inequalities works just like solving a normal equation except when you divide or multiply by a negative number the inequality symbol changes direction. Here are some examples.

Addition and subtraction work.

$$\begin{gathered} 1<2 \\ 1+5<2+5 \\ 6<7 \\ \checkmark \end{gathered}$$

$$\begin{aligned} 1 &< 2 \\ 1-4 &< 2-4 \\ -3 &< -2 \\ &\checkmark \end{aligned}$$

Remember -3 is less than -2 since it is further to the left on a number line. In other words -3 is more negative than -2.

Multiplication and division work as expected with positive numbers.

$$\begin{aligned} 4 &< 6 \\ 4\times 2 &< 6\times 2 \\ 8 &< 12 \\ &\checkmark \end{aligned}$$

$$\begin{aligned} 4 &< 6 \\ 4\div 2 &< 6\div 2 \\ 2 &< 3 \\ &\checkmark \end{aligned}$$

We need to be careful when multiplying and dividing by negative numbers.

$$\begin{aligned} 4 &< 6 \\ 4\times -2 &< 6\times -2 \\ -8 &\not< -12 \\ -8 &> -12 \end{aligned}$$

> **Note**
>
> Remember the following key point when using inequalities:
> When multiplying or dividing by a negative number change the direction of the inequality.

5.3 Simultaneous equations

Sometimes equations have more than one unknown. Take $x+y=4$ for example. There are infinitely many pairs of numbers, x and y, that work for this. Take the following pairs for example: $x=1$ and $y=3$, $x=-100$ and $y=104$, and $x=0.1$ and $y=3.9$.

> **Pro-tip**
>
> These pairs of solutions are often given as co-ordinate pairs like $(1,3)$, $(-100,104)$ and $(0.1,3.9)$. We'll do more about co-ordinates later.

However, if I give you some more information, say $x = y$, now there is only one solution, namely $x = 2$ and $y = 2$. We can use the information in two equations together to find the values that satisfy both equations.

5.3.1 Elimination method

The idea with this method is to combine the two equations to create a new equation with only one variable in it.

$$\begin{aligned} 4x + 2y &= -6 \quad (1) \\ -2x + 3y &= 7 \quad\;\; (2) \end{aligned}$$

To get a solution for y, if we multiply equation (2) by 2 we will have two equations with equal and opposite x-coefficients:

$$\begin{aligned} 4x + 2y &= -6 \\ -4x + 6y &= 14 \quad (3) \end{aligned}$$

If we add equation (1) to equation (3) this eliminates the x-terms, leaving us with one equation in terms of y:

$$\begin{aligned} (2+6)y &= -6 + 14 \\ 8y &= 8 \\ y &= 1 \end{aligned}$$

To obtain a solution for x we can substitute this y-value into either of our initial equations. Using equation (1), we obtain:

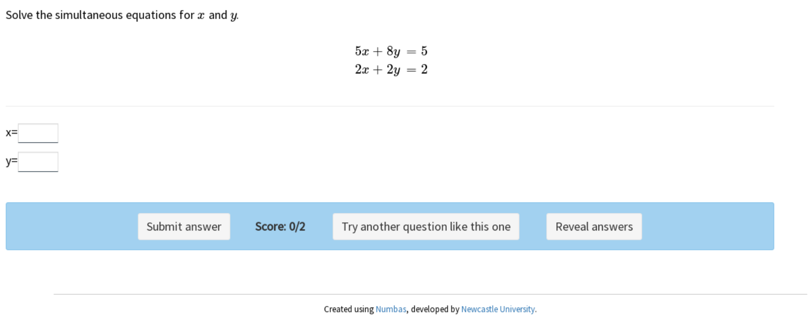
$$\begin{aligned} 4x + 2 \times 1 &= -6 \\ 4x + 2 &= -6 \\ 4x &= -8 \\ x &= -2 \end{aligned}$$

We can check our values for x and y by substituting them into equation 2.

$$
\begin{aligned}
-2x + 3y &= 7 \\
-2 \times -2 + 3 \times 1 &= 7 \\
4 + 3 &= 7
\end{aligned}
$$

Which works out!

You can try other examples in the exercise below. Sometime you may have to multiply both of your starting equations in order to get the same amount of one variable. Also, don't worry if you have eliminated the other variable - it doesn't matter which you get rid of first, you should get the same answer in the end.



5.3.2 Substitution method

It is also possible to re-arrange one equation and substitute it into the other. This method will be covered in the Quadratics section.

6 Reading mathematics

This section looks at common notation used when writing mathematics using formal notation - read it now or come back to it once you've done a bit of *real maths*. You could even use it as a glossary, come back to it and look stuff up if you need to.

Sometimes looking at a piece of mathematics can feel like looking at another language. If you feel that way don't worry, that's normal. It's worth remembering these things:

- **Written mathematics is dense.** A lot of concepts can be expressed with very few symbols. Don't worry if it takes you a while to understand what they mean - that's totally normal. It's also a good idea to get a pen and paper out and *play* with the concepts being expressed.
- **Understanding notation takes time.** At first it can seem unnecessary and needlessly complicated to introduce new symbols. However, once you've mastered using these symbols you will gain a new perspective on the concepts your studying.
- **Practice helps.** Maths is an active subject, take the time to do some questions. Don't be content to read the notes and watch the videos. It's also worth trying to work through examples in your lecture notes alone, even if you've seen the answer before, getting to it yourself will be good practice.

6.1 Common symbols

These symbols can turn up in mathematical explanations.

symbol	meaning
$\therefore$	therefore
$\because$	because
$\neq$	not equal

6.2 Sets

A set is a collection of elements (things). Sets are defined using *curly brackets* or braces $\{$ and $\}$. Capital letters are often used as names of sets. Here is the set of the first 5 multiples of 3 (the first 5 numbers in the 3 times table):

$$A = \{3, 6, 9, 12, 15\}$$

When sets are small it's ok just to write down all the elements of the set. However if I wanted to write down all of the multiples of 3 I would be in trouble. This is when we use **set builder** notation and some new symbols.

$$B = \{3x|x \in \mathbb{N}\}$$

This is read as: *B is the set of* 3 *times x such that x is a natural number.* We've added in a funny E, N and a line! Here's what they mean:

symbol	meaning
$\in$	is a member of the set
$\vert$	such that
$\mathbb{N}$	the natural numbers $1, 2, 3, 4...$

When reading this for the first time it is fine to try some values for x and see what you get. Explore the idea with pen and paper.

6.2.1 Common sets of numbers

The table below contains common sets you may see. Each lower set contains the one above, i.e. the whole of $\mathbb{N}$ is in $\mathbb{Z}$.

symbol	name	example
$\mathbb{N}$	the natural numbers	positive whole numbers $1, 2, 3, 4...$, this sometimes includes zero
$\mathbb{Z}$	the integers	positive and negative whole numbers $..., -2, -1, 0, 1, 2, ...$
$\mathbb{Q}$	the rational numbers	including fractions $-\frac{1}{2}, 0, \frac{1}{2}, 1, ...$
$\mathbb{R}$	the real numbers	now we introduce e and π, numbers with infinite and non-repeating decimal expansions
$\mathbb{C}$	the complex numbers	$\sqrt{-1}$ is now allowed, this enables any polynomial to be solved

Practice with your knowledge of sets with these questions:

a)

Given that $W = \{3, 5, 12\}$

Find the set
$\{4w|w \in W\}$

Enter a list of numbers separated by , .

Submit part

Score: 0/1
Unanswered

b)

Find the set

$\{x \in \mathbb{N}|3 < x \leq 6\}$

Enter a list of numbers separated by , .

Submit part

Score: 0/1
Unanswered

Submit all parts **Score: 0/2** Try another question like this one Reveal answers

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7 Straight line graphs

It is often useful to plot graphs of functions to gain an understanding of what they mean. Straight line graphs are produced by linear equations. Linear equations like $y = 2x + 4$ only have x to the power of one only. Note: this doesn't just apply to x, it could be whatever variable you are using.

7.1 Coordinates

To build a picture of a function we work out pairs of values that satisfy the function. Take for example $y = \frac{1}{2}x + 1$. If we choose values of x we can work out the corresponding y values.

x	y
0	$\frac{1}{2}(0) + 1 = 1$
1	$\frac{1}{2}(1) + 1 = 1.5$
2	$\frac{1}{2}(2) + 1 = 2$

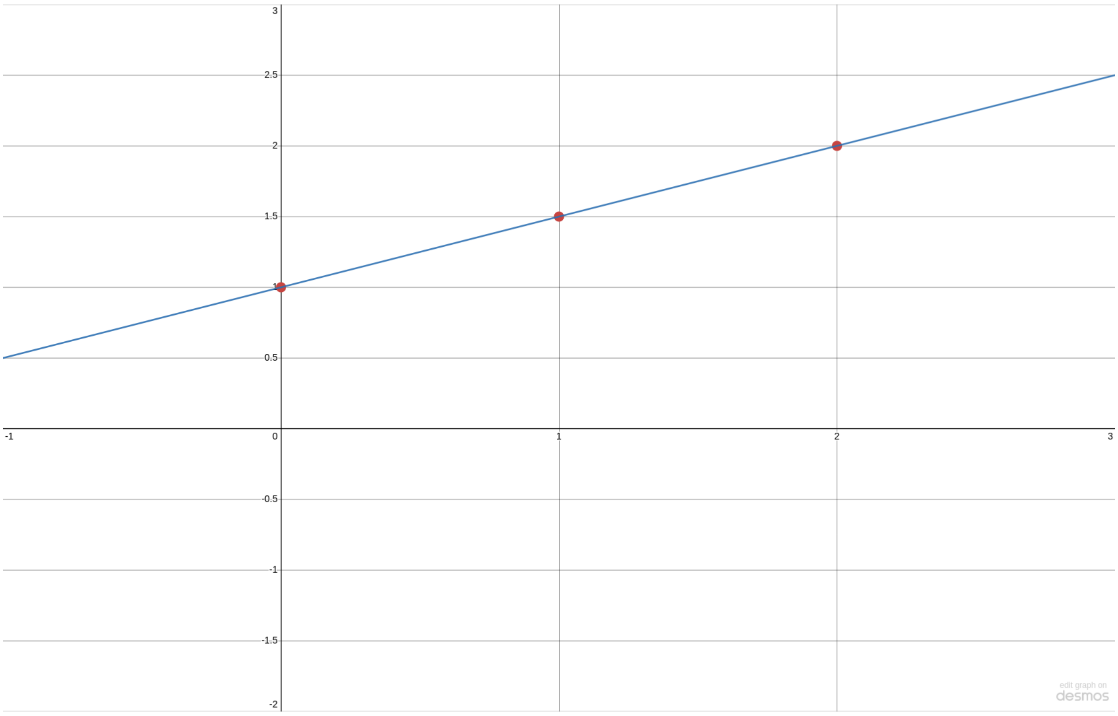
Once we have these values they can be plotted on graph.

The red dots show the points and the blue line shows the equation.

By working out some co-ordinates in the following question try to generate the correct line.

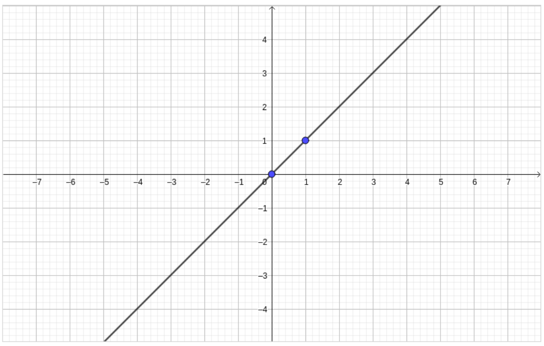
7.2 The formula for a straight line graph: $y = mx + c$

Straight line graphs can be defined by two quantities. The gradient, m, a measure of how steep the line is, and the y intercept, c, where the line crosses the y axis.



Move the points A and B to make line $y = -3x - 3$

You can zoom and pan this image.



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Typesetting math: 100%

7.2.1 The y intercept: c

The y intercept is where line crosses the y axis. We can quickly work out the co-ordinate by substituting $x = 0$ into the equation of a line, or, by noticing the constant term in equation where $y = mx + c$. Here are two examples:

For the line $y = 3x + 4$, the y intercept is at $(0, 4)$ i.e. it crosses the y axis at 4. We can check this by substituting $x = 0$ into the equation.

$$\begin{aligned}
y &= 3x + 4 \\
&= 3(0) + 4 \\
&= 3 \times 0 + 4 \\
y &= 4
\end{aligned}$$

We need to be careful with the next example: $y + 2 = 5x$. It's tempting to say that the y intercept is 2 but it's not. First we must re-arrange the equation into the form of $y = mx + c$. We'll use the idea of doing the same thing to both sides again.

$$\begin{aligned}
y + 2 &= 5x \\
y + 2 - 2 &= 5x - 2 \\
y &= 5x - 2
\end{aligned}$$

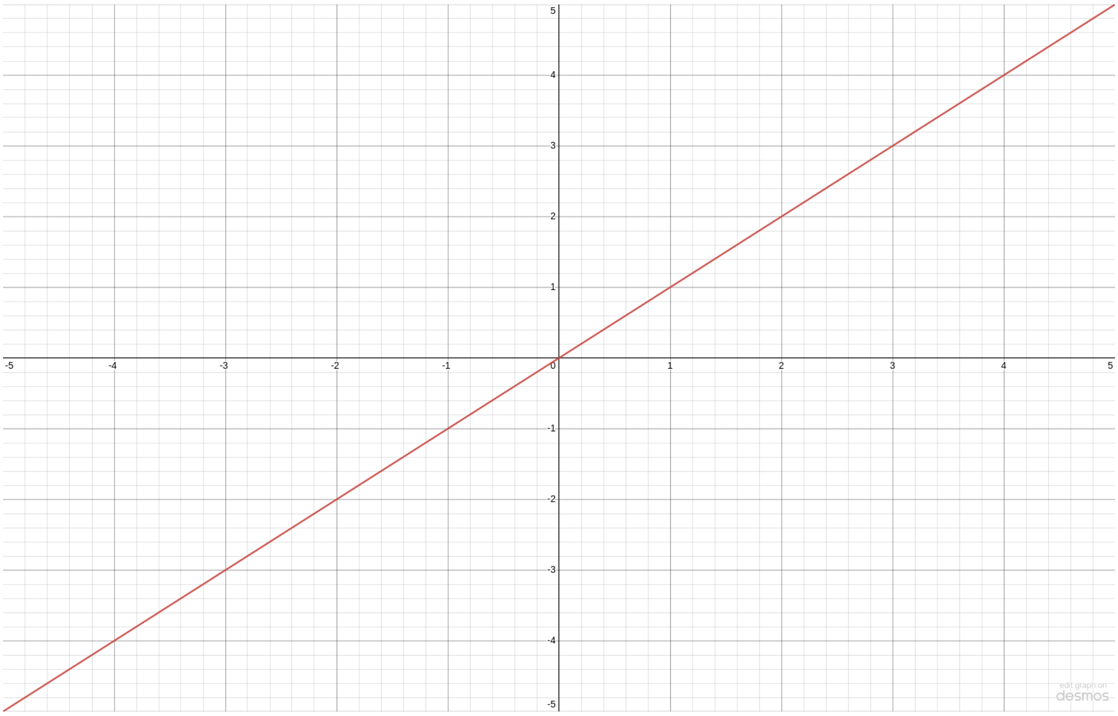
Once we've done this we can see that the intercept is when $y = -2$. Notice if we substituted $x = 0$ in the original equaiton we would get this answer too.

$$\begin{aligned}
y + 2 &= 5x \\
y + 2 &= 5(0) \\
y + 2 &= 0 \\
y &= -2
\end{aligned}$$

Click on the graph below and play with the slider for c. Notice how the graph moves up and down.

7.2.2 The gradient: m

The gradient of a graph is a measure of how much steep the line is. The value of m is the change in the y axis for each increase of 1 in the x axis. So a gradient of $m = 2$ would mean the y values increase by 2 for each increase of 1 in the x direction. This is a positive gradient. Contrast this to a value of m such as -0.5. This means for each increase of 1 in the x direction, the corresponding y value decreases by 0.5 or a half. This is a negative gradient.



The gradient can also be found by calculating the change in the y direction divided by the change in the x direction. The graph below shows how you could calculate the gradient of the line. The line shown has a gradient of $\frac{2}{3}$.

Pro tip

A change in a quantity is often represented by the Greek letter delta, Δ, so we can rewrite m as: $m = \frac{\Delta y}{\Delta x}$

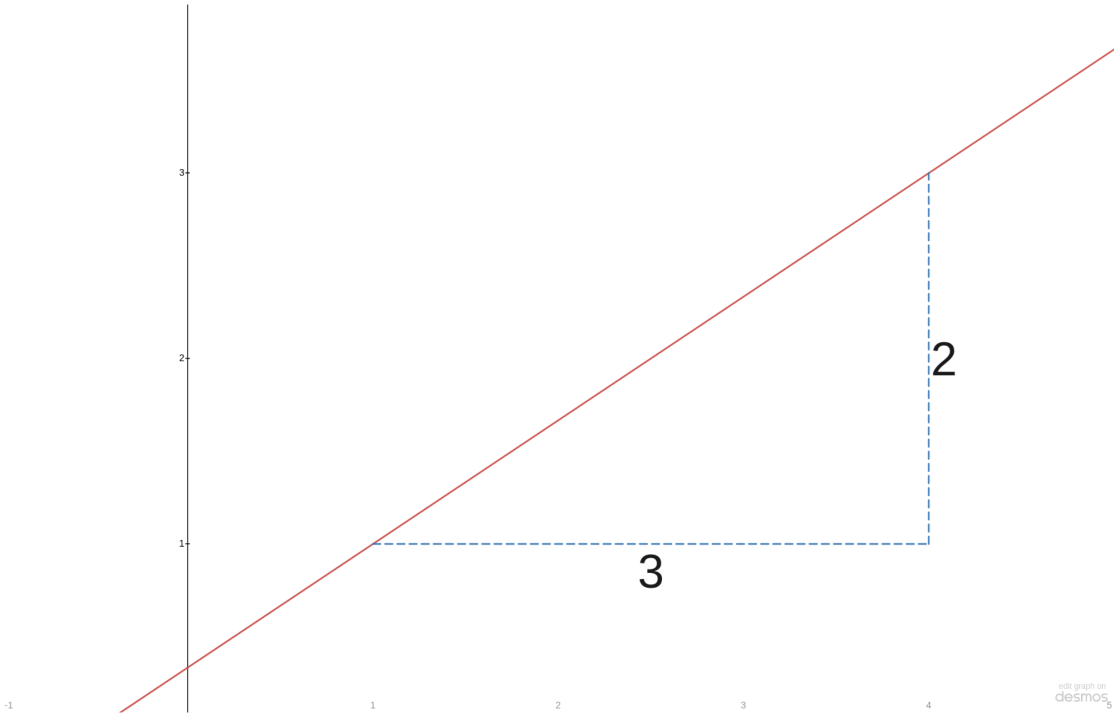
Click on the graph below and then change the value of m with the slider. Notice how the gradient changes but the y intercept stays the same.

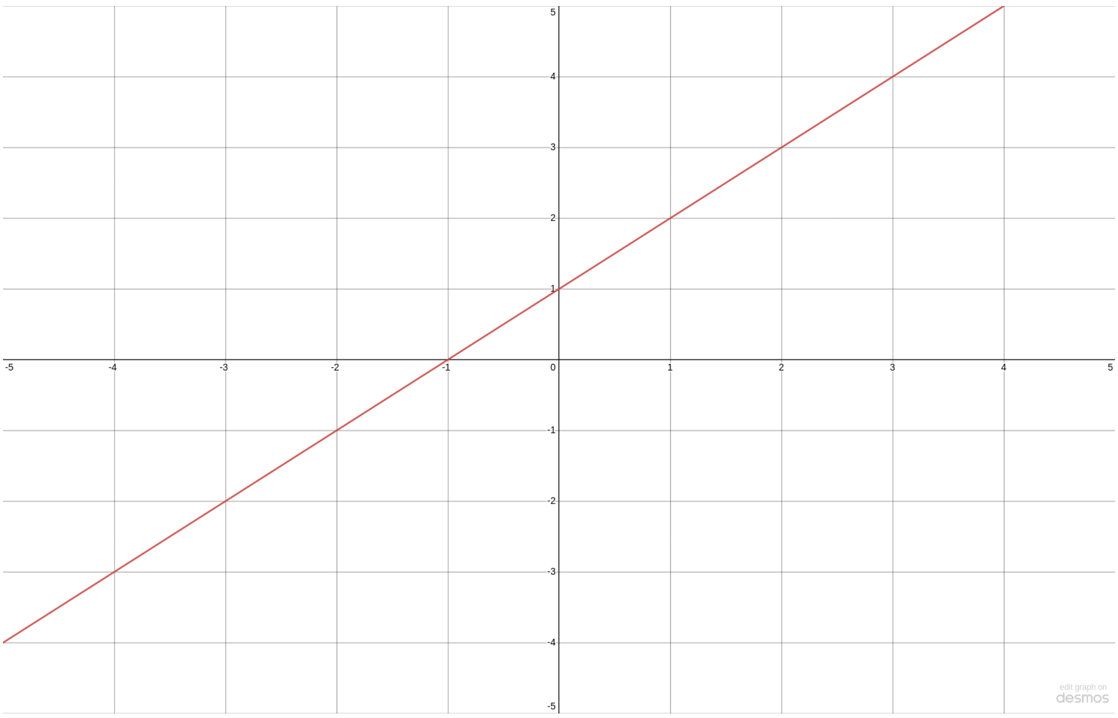
Note

- m is the gradient - the amount y changes for an incease in 1 in the x direction
- c where the line crosses the y axis
- m and c only make sense when the line is in the form $y = mx + c$

Different notation - same thing

The equation of a straight line can be written using different letters. They all mean the same thing. You may see:


2
3
edit graph on
desmos


edit graph on
desmos

- $y = mx + b$
- $y = mx + y_0$
- $y = ax + b$

Using your knowledge of $y = mx + c$ try the following questions. Don't be afraid to look at the answers and then try a fresh set of questions if it seems tricky at first.

$y = \frac{1}{3}x + 2$

a)

Does this line have a positive or negative gradient?

○ Positive ○ Negative

Submit part

Score: 0/1

Unanswered

b)

What is the gradient of this line?

Submit part

Score: 0/1

Unanswered

c)

What is the y-intercept of this line?

This is the point where the function intersects the y-axis.

(0,)

Submit part

Score: 0/1

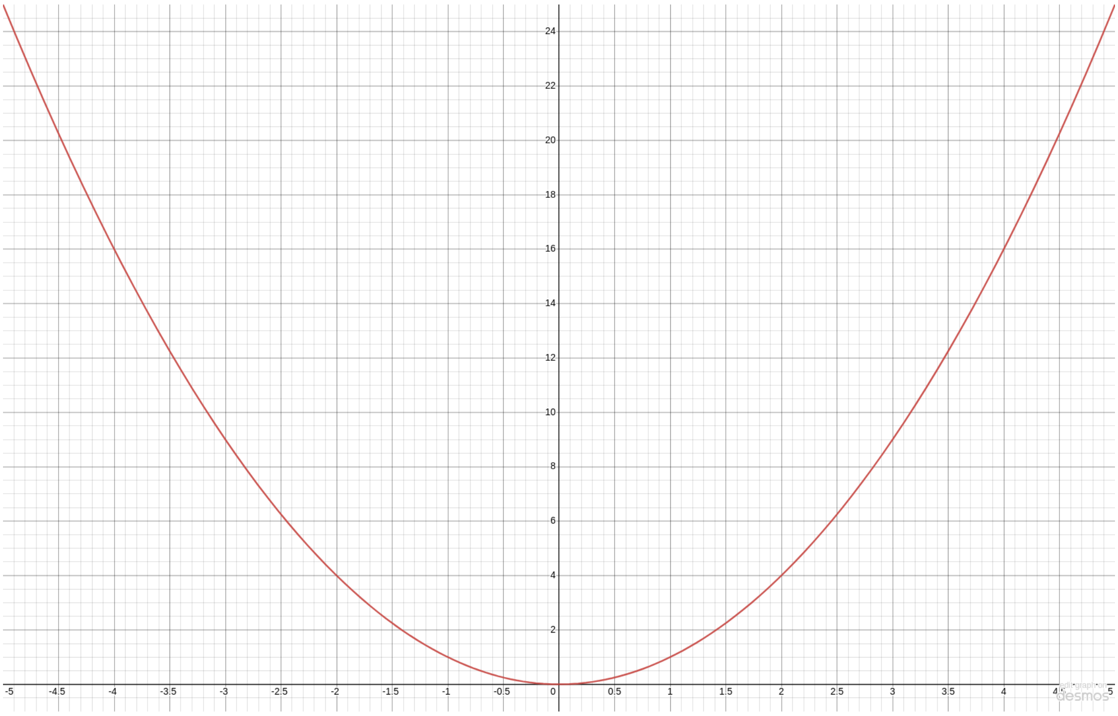
Unanswered

Submit all parts **Score: 0/3** Try another question like this one Reveal answers

8 Quadratics

Quadratics often appear in mathematics, they occur when you have something squared, like x^2. They produce 'U' shaped graphs that can be either way up (depending on the sign of the x^2 term), and, a powerful formula is known that we can use to solve them.

A plot of $y = x^2$ is below:



Quadratics can occur when we expand pairs of brackets, so I've included in this section.

8.1 Expanding pairs of brackets

Expanding a pair of brackets is much the same as a single bracket. However there is a little more going on. Consider this example of a mental method to calculate 25×16.

$$\begin{aligned}25 \times 16 &= (20+5) \times (10+6) \\ &= \overbrace{20 \times 10 + 20 \times 6}^{20\times(10+6)} + \overbrace{5 \times 10 + 5 \times 6}^{5\times(10+6)} \\ &= 200 + 120 + 50 + 30 \\ &= 400\end{aligned}$$

With algebra it works in the same way:

$$\begin{aligned}(a+b)(c+d) &= (a+b) \times (c+d) \\ &= \overbrace{a \times c + a \times d}^{a\times(c+d)} + \overbrace{b \times c + b \times d}^{b\times(c+d)} \\ &= ac + ad + bc + bd\end{aligned}$$

8.2 Factorising pairs of brackets

To factorise a quadratic in the form x^2+bx+c into a pair of brackets like $(x+p)(x+q)$, we look to see if there are a pair of numbers p and q that add to get b and multiply to get c.

$$p + q = b \qquad p \times q = c$$

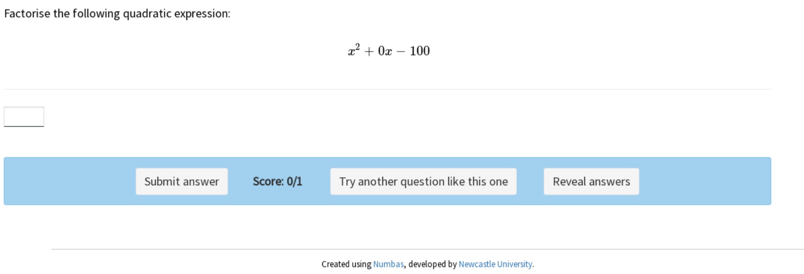
If we can find this pair of numbers we can factorise the quadratic. For example for the quadratic $x^2 + 8x + 12$ we can look at the factors of 12 to help us.

$$\begin{aligned}12 &= 1 \times 12, & 1 + 12 &= 13 \\ 12 &= 2 \times 6, & 2 + 6 &= 8 \\ 12 &= 3 \times 4, & 3 + 4 &= 7\end{aligned}$$

Notice how 2 and 6 multiply to get 12 **and** add to get 8. This means we have the correct pair. So we can now factorise the quadratic:

$$x^2 + 8x + 12 = (x+2)(x+6)$$

Here are some practice questions.



8.3 Solving Quadratics

Interestingly three things can happen when we solve a quadratic. There can be:

- two different values that satisfy the equation
- one *repeated* value
- no real values (only imaginary ones - and yes that is a thing!)

Here are some methods to solve quadratic equations.

8.3.1 Factorisation

We can solve some quadratics by factorisation. Take for example the following equation $x^2 + 8x = -12$. To solve via factorisation we must first make it equal to zero and then factorise. So we have:

$$\begin{aligned} x^2 + 8x &= -12 \\ x^2 + 8x + 12 &= -12 + 12 \\ x^2 + 8x + 12 &= 0 \end{aligned}$$

Now, with a little sense of deja vu (see the example in the previous section) we can factorise our quadratic to get $(x + 2)(x + 6) = 0$. Notice that this is one bracket multiplied by another to get the answer zero. When this happens, i.e. when you multiply two numbers and the answer is zero, either the first number is zero or the second one is. This means either $x + 2 = 0$ or $x + 6 = 0$. Solving these two mini-equations gives the two solutions: either $x = -2$ or $x = -6$.

Pro tip

We can quickly get from the factorised quadratic to the solutions by *flipping* the signs in the bracket.

Try some questions.

Solve the following quadratic equation by factorisation or otherwise:

$$x^2 + 11x + 28 = 0$$

$x =$ ____ *Enter a list of numbers separated by ,.*

Submit answer | **Score: 0/1** | Try another question like this one | Reveal answers

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8.3.2 Quadratic Formula

For a quadratic equation of the form $ax^2 + bx + c = 0$ we can use the quadratic formula to find solutions for x.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We can use the formula on the equation $x^2 - 4x + 2 = 0$. In this example the values of a, b and c are:

- $a = 1$ since x^2 means $1 \times x^2$
- $b = -4$ notice how the negative sign is *owned* by the x coefficient
- $c = 2$ finially we just have 2

Substituting into the quadratic formula we have:

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16 - 8}}{2} \\ &= \frac{4 \pm \sqrt{8}}{2} \end{aligned}$$

It is possible to simplify the square roots in this answer to give $2 \pm \sqrt{2}$. So don't be surprised if your calculator gives you that answer.

Finally, we must deal with the $\pm$ symbol. This means do the calculation once using addition, $+$, and another time using subtraction, $-$. This will give two possible answers for x, given to 2 decimal places.

$$\begin{aligned} x_1 &= \frac{4 + \sqrt{8}}{2} \\ &= 3.41 \end{aligned}$$

and

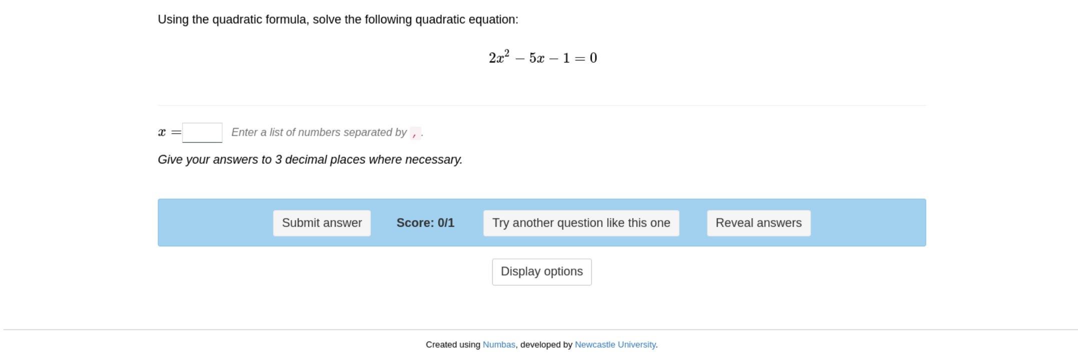
$$\begin{aligned} x_2 &= \frac{4 - \sqrt{8}}{2} \\ &= 0.59 \end{aligned}$$

Pro tip

Notice the use of x_1 and x_2. It is common in maths to use subscript numbers to show different particular values of the same variable. That's all it's doing x_1 is just a value for x named x_1 and x_2 is just a value for x named x_2.

8.4 Simultaneous equations

We are going to solve this type of equation by substitution i.e. substituting one equation into another.



To solve a pair of simultaneous equations of this type we want to rearrange the linear equation such that it is in terms of x or y, which we can then substitute into the equation with the quadratic terms. This will result in a quadratic equation in terms of one variable only.

For the equations:

$$\begin{aligned} 2x + y &= 1 \qquad (1) \\ 3x^2 + 3y^2 &= 4 \qquad (2) \end{aligned}$$

we can rearrange equation (1) to make y the subject:

$$y = 1 - 2x \qquad (3)$$

Substituting equation (3) into equation (2) we have:

$$
\begin{aligned}
3x^2 + 3y^2 &= 4 \\
3x^2 + 3(1-2x)^2 &= 4 \\
3x^2 + 3(1-2x)(1-2x) &= 4 \\
3x^2 + 3(1-4x+4x^2) &= 4 \\
3x^2 + 3 - 12x + 12x^2 &= 4 \\
15x^2 - 12x - 1 &= 0
\end{aligned}
$$

Warning

There are a few things to be careful of here:

- $(1-2x)^2$ was expanded as a pair of brackets, $(1-2x)(1-2x)$ before being multiplied by 3.
- The finial stage was to make the equation equal zero so we can use the quadratic formula.

Now we have an equation we can solve we can use the quadratic formula. To find values of x. This gives two solutions $x_1 = -0.08$ to 2 decimal places, and, $x_2 = -0.88$ again to 2 decimal places.

Finally, since our equations for x and y we need to find corresponding y values for each x. The easiest way to do this is to use equation (3). This gives, $y_1 = 1.15$ and $y_2 = -0.75$. Note, to maintain accuracy you'll need to put your *full* values for x_1 and x_2 into equation (3) and then round to 2 decimal places afterwards.

This gives two pairs of numbers for our answer. $(x_1, y_1) = (-0.08, 1.15)$ and $(x_2, y_2) = (0.88, -0.75)$.

Pro tip

notice our answers look a lot like co-ordinates on a graph. That's because they are. If you plot the lines $2x + y = 1$ and $3x^2 + 3y^2 = 4$ on the same graph (don't do this by hand! Use something like desmos) the places where the two lines cross will correspond with our answers.

Here are some practice questions. Don't forget you can graph them if it helps.

Solve the following simultaneous equations:

$$\begin{aligned} 4x + y &= 2 \\ 2x^2 + 5y^2 &= 49 \end{aligned}$$

Give your answers to 2 decimal places where necessary.

$(x_1, y_1) = \left(\square \quad \square \right)$

$(x_2, y_2) = \left(\square \quad \square \right)$

Submit answer **Score: 0/2** Try another question like this one Reveal answers

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9 Indices

Indices is another word for powers. In this section we move beyond the idea that powers are just repeated multiplications.

9.1 Index notation

Being comfortable moving between different ways to write powers helps when rearranging algebra.

Note

- $x^0 = 1$ except when $x = 0$ then it's undefined
- $x^{-n} = \frac{1}{x^n}$
- $x^{\frac{1}{n}} = \sqrt[n]{x}$

Here are some examples:

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

More generally.

$$x^{-3} = \frac{1}{x^3}$$

Anything to the power of zero is 1:

$$\pi^0 = 1$$

Remember good old π? From working stuff out about circles $\pi = 3.14159...$

We can write square roots:

$$16^{\frac{1}{2}} = \sqrt{16} = \pm 4$$

Pro tip

When taking square roots remember there are two possible solutions. Since in the above example $4 \times 4 = 16$ and $-4 \times -4 = 16$. So either answer is just fine.

Here's an example of a cube root.

$$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$$

9.1.1 Combinations, roots and powers

A roots and powers can be combined. If a number is raised to the power of a fraction you find the root corresponding to the denominator and then raise it to the power of the numerator. For example:

$$8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = (2)^2 = 4$$

Cube root, because of the 3 in the denominator, then square the answer because of the 2 in the numerator. This sequence could be done the other way around, square first then cube root, I choose this way since the numbers stay smaller.

9.1.2 Reciprocals

If you raise a number to the power of -1 you find it's reciprocal (you **flip it**). For example:

$$\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$$

9.1.3 But why?

Just like we did with negative numbers we can extend the idea of what a power means by following a pattern. Here's a pattern to justify $x^0 = 1$ and $x^{-n} = \frac{1}{x^n}$.

$$\begin{aligned}
10^3 &= 10 \times 10 \times 10 &&= 1000 \\
10^2 &= 10 \times 10 &&= 100 \\
10^1 &= 10 &&= 10 \\
10^0 &= 1 &&= 1 \\
10^{-1} &= \frac{1}{10} &&= 0.1 \\
10^{-2} &= \frac{1}{10 \times 10} &&= 0.01 \\
10^{-3} &= \frac{1}{10 \times 10 \times 10} &&= 0.001
\end{aligned}$$

I'll come back to the justification about square roots after the next section.

9.2 Rules of indices

There is a neat set of rules we can use when combining numbers with indices:

> **Note**
>
> - $x^n \times x^m = x^{n+m}$
> - $x^n \div x^m = x^{n-m}$
> - $(x^n)^m = x^{n \times m}$

When you multiply terms you add the powers.

$$\begin{aligned} 3x^4 \times 5x^6 &= 3 \times 5 \times x^4 \times x^5 \\ &= 15 \times x^{4+5} \\ &= 15x^9 \end{aligned}$$

Lets put it all together with a complicated example:

To rewrite $\frac{\sqrt[4]{x^5x^3}}{\sqrt[3]{x}\sqrt[6]{x^3}}$ in the form x^n, we need to use the following rules:

1. $a^n a^m = a^{n+m}$;
2. $\sqrt[n]{a} = a^{1/n}$;
3. $(a^n)^m = a^{n \times m}$;
4. $\frac{a^n}{a^m} = a^{n-m}$.

We will simplify the numerator and denominator separately to make the steps clearer. Firstly, applying rule 1, then rule 2, and then rule 3 to the numerator:

$$\begin{aligned} \frac{\sqrt[4]{x^5x^3}}{\sqrt[3]{x}\sqrt[6]{x^3}} &= \frac{\sqrt[4]{x^8}}{\sqrt[3]{x}\sqrt[6]{x^3}} \\ &= \frac{(x^8)^{1/4}}{\sqrt[3]{x}\sqrt[6]{x^3}} \\ &= \frac{x^2}{\sqrt[3]{x}\sqrt[6]{x^3}} \end{aligned}$$

To simplify the denominator, we want to apply rule 2, then rule 3, and then rule 1:

$$\begin{aligned}
\frac{x^2}{\sqrt[3]{x}\sqrt[6]{x^3}} &= \frac{x^2}{x^{1/3}(x^3)^{1/6}} \\
&= \frac{x^2}{x^{1/3}x^{1/2}} \\
&= \frac{x^2}{x^{5/6}}
\end{aligned}$$

Remember that we'll need to get common denominators when adding the fractions at the end:

$$\begin{aligned}
\frac{1}{3}+\frac{1}{2} &= \frac{1\times 2}{3\times 2}+\frac{1\times 3}{2\times 3} \\
&= \frac{2}{6}+\frac{3}{6} \\
&= \frac{5}{6}
\end{aligned}$$

Finally, applying rule 4 and simplifying,

$$\begin{aligned}
\frac{x^2}{x^{5/6}} &= x^2\times x^{-5/6} \\
&= x^{2-5/6} \\
&= x^{12/6-5/6} \\
&= x^{7/6}
\end{aligned}$$

Lots of work with fractions here!

Now try these questions. Don't worry if it takes a while to just solve one!

9.2.1 But why? Square roots

As promised here is an explanation of why $x^{\frac{1}{n}} = \sqrt[n]{x}$.

When we take a square root we look for the a number that when it is multiplied by it's self we get the answer i.e. $? \times ? = x$. Since one x is the same as x^1 we can rewrite out statement again:

$$\begin{aligned}
? \times ? &= x^1 \\
x^? \times x^? &= x^1 \\
x^{?+?} &= x^1
\end{aligned}$$

This means $?+? = 1$ so $? = \frac{1}{2}$ so $x^{\frac{1}{2}} = \sqrt{x}$.

Rewrite the following expression as a single term, in the form x^n, where n is a fraction.

$$\frac{\sqrt[5]{x^3 x^7}}{\sqrt[3]{x}\,\sqrt[6]{x^3}}$$

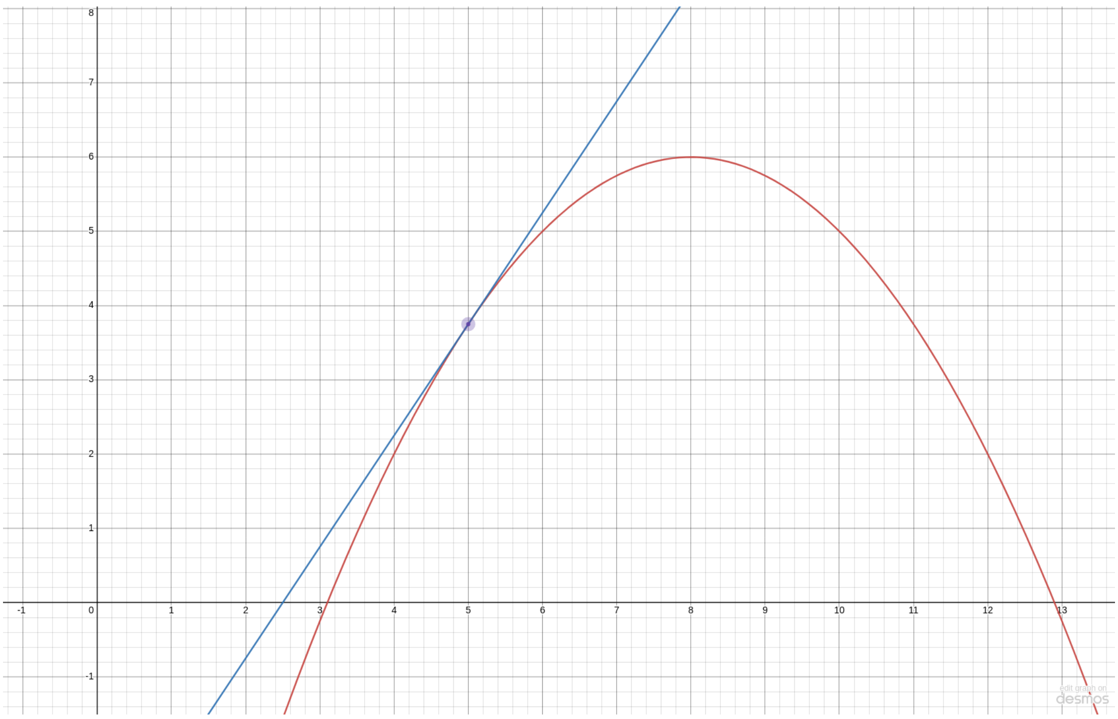
Submit answer Score: 0/1 Try another question like this one Reveal answers

10 Differentiation

We often want to be able to find the gradient of a curved line. For that we need a new technique, called differentiation, that will give us a rule (a new function) to work out the gradient at any point on the curve.

10.1 The tangent to a curve

The gradient at a point on a curve is the same as the gradient of the tangent at that point. A tangent to a curve is a straight line that just touches curve at that point. Below is a picture of the tangent to the curve when $x = 5$. You can open up the graph and move the point around with the slider.



Notice that the gradient will change depending on which value of x you use.

10.2 The rules of differentiation

Luckily finding the rule to get the gradient of a curve is straight forward. The language we use for this process is like this. When function is differentiated a new function, the derivative, is found. The derivative enables you to find the gradient. There are lots of ways write this in mathematical notation. Here are the most common.

original function	derivative
y	$\frac{dy}{dx}$
$f(x)$	$f'(x)$

$\frac{dy}{dx}$ is pronounced 'dee y by dee x', and $f'(x)$ is read as 'f dash of x'.

The rule for differentiating polynomials (functions made up of adding different powers of x)is:

> **Note**
>
> - if $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$, or,
> - if $f(x) = ax^n$ then $f'(x) = anx^{n-1}$ **Times by the power, then take one off the power**

Here are some examples:

If $y = 3x^4$ then $\frac{dy}{dx} = 3 \times 4 \times x^{4-1} = 12x^3$

Multiple terms added together are differentiated one by one then added together:

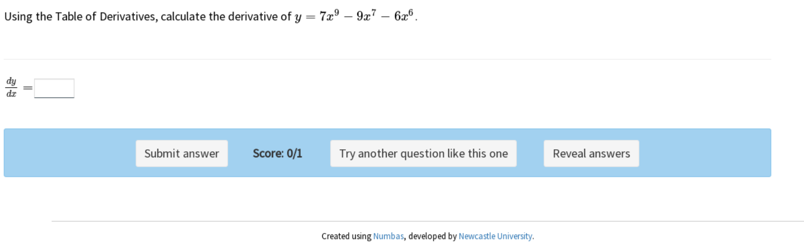
$$\begin{aligned}
y &= 6x^3 + 2x^2 + 4x + 5 \\
\frac{dy}{dx} &= 6x^3 + 2x^2 + 4x^1 + 5x^0 \\
&= 3 \times 6x^{3-1} + 2 \times x^{2-1} + 1 \times 4x^{1-1} + 0 \times 5x^{0-1} \\
&= 18x^2 + 2x^1 + 4x^0 + 0 \\
&= 18x^2 + 2x + 4
\end{aligned}$$

In the above example we've used the following mathematical facts:

- $x = x^1$, x on it's own is x^1
- $x^0 = 1$, you can always multiply by x^0 since it's 1
- $0 \times a = 0$ anything times zero is zero

The take away from this is that constant terms, terms without x in, disappear, and terms with just x in loose the x.

Try these questions to get to grips with the rules of differentiation.



10.3 Finding gradient at a point

To find the gradient at a point. Differentiate the original function and then substitute the x value of the point into the derivative.

For example to find the gradient when $x = 3$ for the function $y = x^2$. We would differentiate and then substitute in $x = 3$.

$$
\begin{aligned}
y &= x^2 \\
\frac{dy}{dx} &= 2x \\
&= 2(3) \\
&= 2 \times 3 \\
&= 6
\end{aligned}
$$

So the gradient at $x = 3$ on the curve $y = x^2$ is 6.

11 Exponential functions

Exponential functions crop up in applied mathematics everywhere. This section looks at these important functions, so important that, Professor Albert Bartlett said the following about them in this lecture Arithmetic, Population and Energy.

The greatest shortcoming of the human race is our inability to understand the exponential function.

11.1 Getting to know exponential functions

An exponential function comes in the for, $y = a^x$. They can increase incredibly fast. Take for example $y = 2^x$

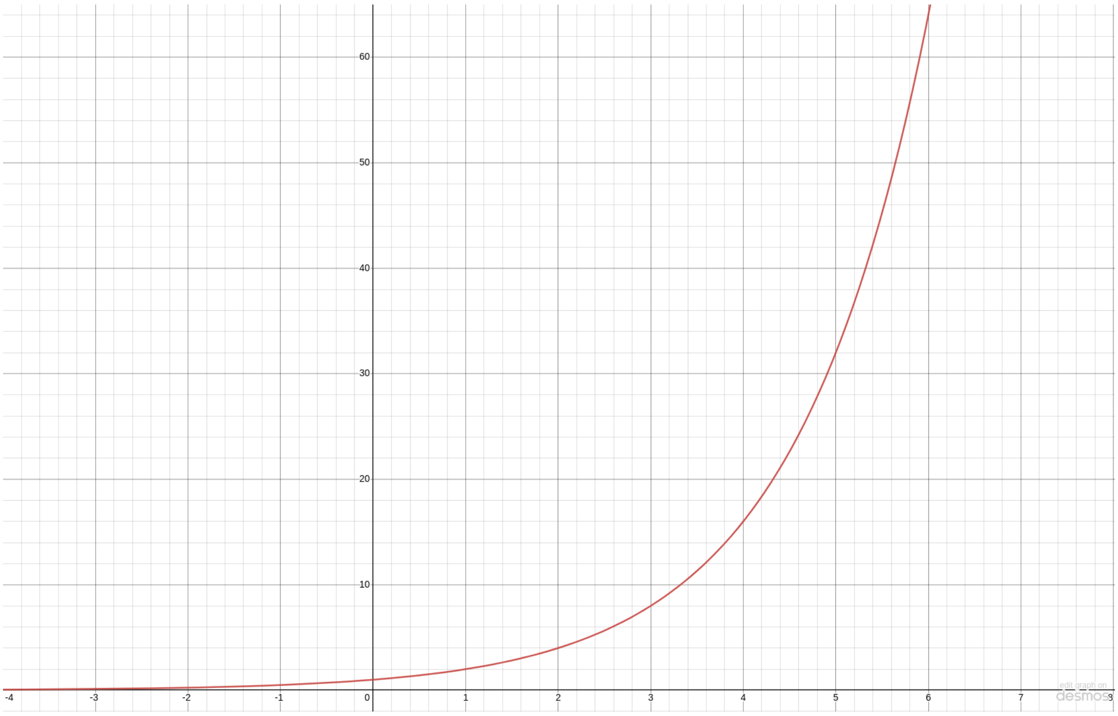
x	$y = 2^x$
-2	$y = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
-1	$y = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
0	$y = 2^0 = 1$
1	$y = 2^1 = 2$
2	$y = 2^2 = 4$
3	$y = 2^3 = 8$
4	$y = 2^4 = 16$

Plotting these points give a graph that looks like:

Notice the following key points about the graph.

Note

- The graph quickly increases.
- It crosses the y axis at 1 (all exponential graphs do this).
- It never goes under the x axis.



11.2 The exponential function

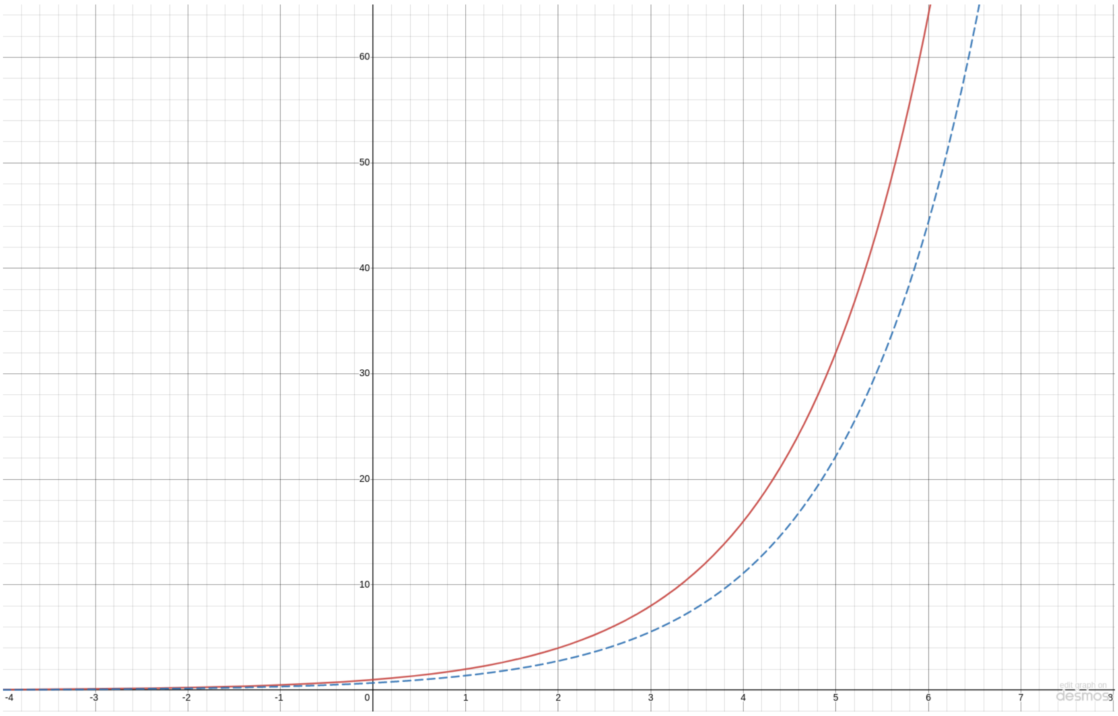
There is one exponential function that is so important that it is called **the** exponential function. It is written as $y = e^x$ where e is an irrational number (an infinitely long decimal number that doesn't repeat itself, $/pi$ is an irrational number too). The value of e is:

$$e = 2.71828182845904523536028747135266249775724709369995...$$

ish.

The reason why it is special is that when $y = e^x$, the derivative is itself, that is $\frac{dy}{dx} = e^x$. Below is a graph of $y = a^x$ (solid red line) and it derivative (dashed blue line), you can open it up and change the value of a from 2 to 4. a is set to 2 to begin with, notice how the derivative is beneath the curve $y = a^x$. When a is increased the derivative moves above $y = a^x$. The point where the two curves overlap is when $a = e$.

> **Note**
>
> If $y = e^x$ then $\frac{dy}{dx} = e^x$.



11.3 Differentiating e^x

The rule for differentiating e^x is if $y = ke^{ax}$ then $\frac{dy}{dx} = ake^{ax}$.

Use that rule to try the following questions.

Calculate the derivative of $y = 9e^{8x}$.

$\frac{dy}{dx} =$

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12 Logarithms

Logarithms, or logs for short, are the same as powers just written in another way.

12.1 Reverse of indices

Key point:

If $a^y = x$ then $y = \log_a x$.

a is called the base of the logarithm. When dealing with logs it's often useful to think of a numerical example to keep the idea straight in your head.

$$10^3 = 1000$$
$$3 = \log_{10} 1000$$

This is the same fact written in index notation and as a logarithm.

12.2 Rules of logarithms

Just as there are rules when dealing with indices, there are the corresponding rules when dealing with logarithms too.

Key point:

- $\log_a x + \log_a y = \log_a xy$
- $\log_a x - \log_a y = \log_a \frac{x}{y}$
- $\log_a x^n = n \log_a x$

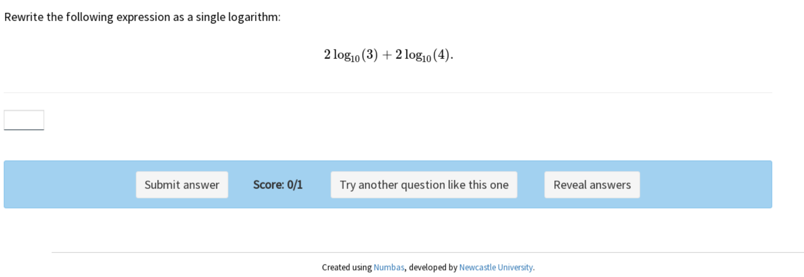
We can use these rules to manipulate algebraic expressions. For example, let's write the following as a single logarithm:

$$
\begin{aligned}
3\log_{10} 2 + \log_{10} 5 - \log_{10} 4 &= \log_{10} 2^3 + \log_{10} 5 - \log_{10} 4 \\
&= \log_{10} 8 + \log_{10} 5 - \log_{10} 4 \\
&= \log_{10}(8 \times 5) - \log_{10} 4 \\
&= \log_{10} 40 - \log_{10} 4 \\
&= \log_{10}(\frac{40}{4}) \\
&= \log_{10}(10) \\
&= 1
\end{aligned}
$$

This is how it was done:

- First we used the power rule $\log_a x^n = n\log_a x$,
- then the addition rule $\log_a x + \log_a y = \log_a xy$,
- and finally, the subtraction rule $\log_a x - \log_a y = \log_a \frac{x}{y}$.
- Then notice $\log_{10}(10) = 1$ since $10^1 = 10$.

Have a go at these simplification questions.



12.3 Solving equations with logarithms in

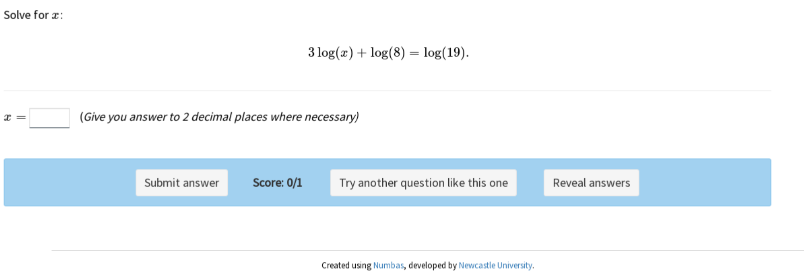
For example, let's solve $3\log_{10}x + \log_{10}2 = \log_{10}250$. First we'll apply the power rule $\log_a x^n = n\log_a x$, then the addition rule $\log_a x + \log_a y = \log_a xy$:

$$
\begin{aligned}
3\log_{10}x + \log_{10}2 &= \log_{10}250\\
\log_{10}x^3 + \log_{10}2 &= \log_{10}250\\
\log_{10}2x^3 &= \log_{10}250
\end{aligned}
$$

Now since the two sides are equal the values inside the logarithm must be equal. We can then go ahead and solve the resulting equation as normal.

$$
\begin{aligned}
\log_{10}2x^3 &= \log_{10}250\\
2x^3 &= 250\\
x^3 &= 125\\
x &= \sqrt[3]{125}\\
&= 5
\end{aligned}
$$

Have a go at the following questions:



12.4 Some important bases

Some bases in logarithms come up more than others, because of that some bases have their own notation.

12.4.1 The natural logarithm

A logarithm that has e as it's base is known as the natural logarithm and has it's own symbol.

Key point:

$$\log_e x = \ln x$$

12.4.2 Base 10

A logarithm that has 10 as it's base has it's own symbol.

Key point:

$$\log_{10} x = \log x$$

You just don't bother writing the base.

12.5 Differentiating $\ln x$

The rule for differentiating $\ln x$ is:

Key point:

if $y = k \ln ax$ then $\frac{dy}{dx} = \frac{k}{x}$.

Use that rule to try the following questions.

Calculate the derivative of $y = 17\ln(7x)$.

$\frac{dy}{dx} =$ ______

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13 Further differentiation

So far we have looked at differentiating powers of x when they are added together. This section introduces differentiating e^x and $\ln x$, then goes on to look at how to differentiate, functions inside functions, products of functions (when functions are multiplied together) and quotients of functions (when functions are divided by each other).

13.1 Standard results

We can now expand our table of derivatives. Here are all the rules from the last differentiation along with some new ones.

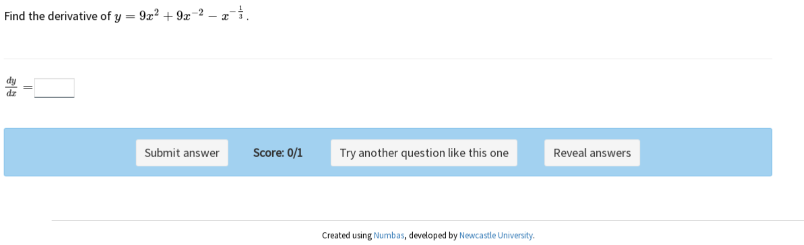
original function	derivative
y	$\frac{dy}{dx}$
$f(x)$	$f'(x)$
$f(x)+g(x)$	$f'(x)+g'(x)$
ax^n	anx^{n-1}
e^x	e^x
e^{ax}	ae^{ax}
$\ln x$	$\frac{1}{x}$
$\ln ax$	$\frac{1}{x}$

We can now happily just apply the rules (and some rules of indices for good measure). For example:

$$\begin{aligned} y &= 2x^4 + e^{2x} + \ln x + \sqrt{x} + 100 \\ &= 2x^4 + e^{2x} + \ln x + x^{1/2} + 100 \\ \frac{dy}{dx} &= 8x^3 + 2e^{2x} + \frac{1}{x} + \frac{1}{2}x^{-1/2} \end{aligned}$$

Notice that $\sqrt{x}$ was rewritten as $x^{1/2}$ to be able to apply the rule ax^n goes to anx^{n-1}.

Try some differentiation with some fractional powers:



13.2 The chain rule

The chain rule is used when we have functions inside other functions.

If we have a function of the form $y = f(g(x))$, sometimes described as a function of a function, to calculate its derivative we need to use the chain rule:

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

This can be split up into steps:

Let $u = g(x)$; Rewrite y in terms of u, such that $y = f(u)$; Calculate $\frac{du}{dx}$ and $\frac{dy}{du}$; Write $\frac{dy}{dx}$ as a product of $\frac{du}{dx}$ and $\frac{dy}{du}$; Make sure $\frac{dy}{dx}$ is only in terms of x. Ensure any u terms have been replaced using the initial substitution.

Following this process, we must first identify $g(x)$. Since the function is of the form $y = f(g(x))$, we are looking for the 'inner' function.

So, for $y = -(4x^2 + 1)^4$,

$$g(x) = 4x^2 + 1.$$

If we now set $u = g(x)$, we can rewrite y in terms of u such that $y = f(u)$:

$$y = -u^4$$

Next, we calculate the two derivatives $\frac{du}{dx}$ and $\frac{dy}{du}$:

$$\frac{du}{dx} = 8x, \qquad \frac{dy}{du} = -4u^3$$

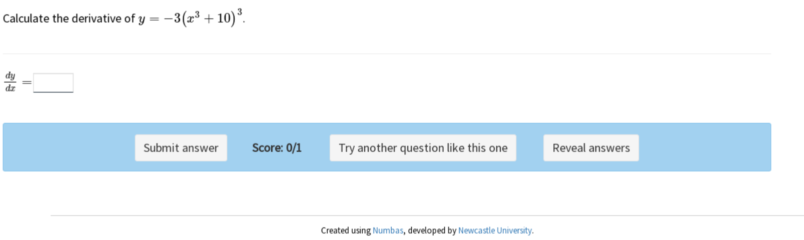
Plugging these into the chain rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx} \times \frac{dy}{du}, \\ &= 8x \times -4u^3, \\ &= -32xu^3. \end{aligned}$$

Finally, we need to express $\frac{dy}{dx}$ only in terms of x, so we must replace the u term using the initial substitution $u = 4x^2 + 1$:

$$\frac{dy}{dx} = -32x(4x^2 + 1)^3.$$

Phew! Time for a cup of tea, or maybe some more questions...



13.3 The product rule

If we have a function of the form $y = u(x)v(x)$, to calculate its derivative we need to use the product rule:

$$\frac{dy}{dx} = u(x) \times \frac{dv}{dx} + v(x) \times \frac{du}{dx}.$$

This can be split up into steps:

Identify the functions $u(x)$ and $v(x)$; Calculate their derivatives $\frac{du}{dx}$ and $\frac{dv}{dx}$; Substitute these into the formula for the product rule to obtain an expression for $\frac{dy}{dx}$; Simplify $\frac{dy}{dx}$ where possible.

Following this process, we must first identify $u(x)$ and $v(x)$.

As

$$y = e^x ln(6x),$$

let

$$u(x) = e^x \quad \text{and} \quad v(x) = ln(6x).$$

Next, we need to find the derivatives, $\frac{du}{dx}$ and $\frac{dv}{dx}$:

$$\frac{du}{dx} = e^x \quad \text{and} \quad \frac{dv}{dx} = \frac{1}{x}.$$

Substituting these results into the product rule formula we can obtain an expression for $\frac{dy}{dx}$:

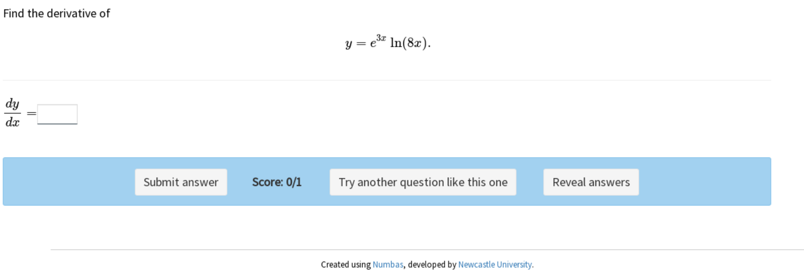
$$\frac{dy}{dx} = \frac{du}{dx} \times v(x) + u(x) \times \frac{dv}{dx}$$

$$= e^x \times \ln(6x) + e^x \times \frac{1}{x}.$$

Simplifying,

$$\begin{aligned}\frac{dy}{dx} &= e^x \ln(6x) + e^x \frac{1}{x} \\ &= e^x(\ln(6x) + \frac{1}{x}).\end{aligned}$$

Now your turn...



13.4 The quotient rule

If we have a function of the form $y = \frac{u(x)}{v(x)}$, to calculate its derivative we need to use the quotient rule:

$$\frac{dy}{dx} = \frac{v(x) \times \frac{du}{dx} - u(x) \times \frac{dv}{dx}}{[v(x)]^2}.$$

This can be split up into steps:

Identify the functions $u(x)$ and $v(x)$; Calculate their derivatives $\frac{du}{dx}$ and $\frac{dv}{dx}$; Substitute these into the formula for the quotient rule to obtain an expression for $\frac{dy}{dx}$; Simplify $\frac{dy}{dx}$ where possible.

Following this process, we must first identify $u(x)$ and $v(x)$.

As

$$y = \frac{e^{2x}}{3x^2 + 4x + 5},$$

let

$$u(x) = e^{2x} \quad \text{and} \quad v(x) = 3x^2 + 4x + 5.$$

Next, we need to find the derivatives, $\frac{du}{dx}$ and $\frac{dv}{dx}$:

$$\frac{du}{dx} = 2e^{2x} \quad \text{and} \quad \frac{dv}{dx} = 6x + 4.$$

Substituting these results into the quotient rule formula we can obtain an expression for $\frac{dy}{dx}$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{v(x) \times \frac{du}{dx} - u(x) \times \frac{dv}{dx}}{[v(x)]^2} \\
&= \frac{(3x^2 + 4x + 5) \times 2e^{2x} - e^{2x} \times (6x + 4)}{(3x^2 + 4x + 5)^2}.
\end{aligned}$$

Simplifying,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{2e^{2x}(3x^2 + 4x + 5) - e^{2x}(6x + 4)}{(3x^2 + 4x + 5)^2} \\
&= \frac{e^{2x}[(6x^2 + 8x + 10) - (6x + 4)]}{(3x^2 + 4x + 5)^2} \\
&= \frac{e^{2x}(6x^2 + 8x + 10 - 6x - 4)}{(3x^2 + 4x + 5)^2} \\
&= \frac{e^{2x}(6x^2 + 2x + 6)}{(3x^2 + 4x + 5)^2} \\
&= \frac{2e^{2x}(3x^2 + x + 3)}{(3x^2 + 4x + 5)^2}.
\end{aligned}$$

Now have a go at these:

Find the derivative of

$$y = \frac{e^{5x}}{2x^2 - 3x + 2}.$$

$\frac{dy}{dx} =$

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